9.1 - Euler Methods and Error Analysis

Definition: Because Euler's method, which uses the iterative formula $y_{n+1} = y_n + hf(x_n, y_n),\ n = 0, 1, 2, \ldots$ to approximate solutions to initial-value problems of the form $y' = f(x, y),\ y(x_0) = y_0$, uses approximations, the y-values obtained this way contain **local truncation error** at each step in the approximation.

To determine an upper bound for the size of the error, consider the following. Suppose a function $y(x)$ possesses $k + 1$ derivatives that are continuous on an open interval containing a and x. The k^{th}-degree Taylor polynomial with remainder for $y(x)$ is

$$y(x) = y(a) + y'(a)\frac{x-a}{1!} + \cdots + y^{(k)}(a)\frac{(x-a)^k}{k!} + y^{(k+1)}(c)\frac{(x-a)^{k+1}}{(k+1)!}$$

where c is some value between a and x. For $k = 1$, $a = x_0$, and $x = x_{n+1} = x_n + h$, we have

$$y(x_{n+1}) = y(x_0) + \frac{y'(x_n)}{1!} + y''(c)\frac{h^2}{2!}$$

This is precisely the formula for Euler's method plus a remainder term. Thus, the local truncation error in y_{n+1} is $y''(c)\frac{h^2}{2!}$, where $x_n < c < x_{n+1}$. Since we typically don't know the value of c, an upper bound for the absolute value of the error is $Mh^2/2$, where $M \geq |y''(x)|,\ x_n < x < x_{n+1}$. We can use the notation $O(h^2)$ to indicate that the magnitude of the error depends on h^2. (This is called "big oh" notation, and it essentially indicates that one quantity is bounded by another quantity.)

Definition: The total error in approximating y_{n+1} contains local truncation error at each step, and so the result after multiple iterations contains **global truncation error**.

It can be shown that global truncation error using Euler's method is $O(h)$.

The improved Euler's method uses an average of two slopes and has a local truncation error $O(h^3)$ and global truncation error $O(h^2)$.

Improved Euler's method

First, we use Euler's method to obtain $y^*_{n+1} = y_n + hf(x_n, y_n)$. This is a predictor step.

Then we use $y_{n+1} = y_n + h\frac{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})}{2}$. This is a corrector step.

Because of the two-step process, this is called a predictor-corrector method.

Example: Use the improved Euler's method to obtain a four-decimal approximation of the indicated value. First use $h = 0.1$ and then use $h = 0.05$.

$y' = 4x - 2y, \qquad y(0) = 2; \qquad y(0.5)$